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ON THE CONVERGENCE OF STOCHASTIC APPROXIMATION PROCEDURES UNDER MARKOV NOISE IN THE MEASUREMENTS*

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The problem is considered of the convergence of stochastic approximation procedures /1,2/ for seeking the zero of a function under the condition that the values of this function, accessible to measurement, contain both external as well as internal perturbations. The statement of this problem differs from the most prevalent ones in that the assumptions on the independence and additivity of the noise are waived. The proof of the convergence is based on the use of stochastic Liapunov functions /3—6/. Discrete stochastic approximation procedures under dependent measurements were examined, for instance, in /7,8/ with another way of accounting for the perturbations and by other methods. The majority of papers on the study of stochastic programming /9/ and stochastic approximation procedures assume the independence of the measurements and the additivity of the noise. Without disparaging such an approach, it should be emphasized that it does not exhaust all varieties of problems whose study might lead to stochastic approximation procedures. In particular, if the measurements are made sufficiently often or, even more so, continuously, then the assumption of dependence of the measurements proves to be very natural, specially if the noise realize parametric perturbations of the system. Other examples, not covered in the scheme of independent measurements, are the problems of adaptive control, of observation, of estimation /10/. There are comparatively few papers (see /7,8/, for example) where the convergence of gradient procedures of extremum search is proved in the presence of additive Markov noise. Conditions are formulated in the present paper on the convergence of stochastic approximation procedures under the condition that the measurements contain both additive as well as nonadditive (internal) Markov perturbations. The analysis is restricted to procedures of the Robbins-Monroe type, mainly in the continuous version.

1. Statement of the problem. Let $f(x)$ be an unknown n-vector-valued function defined on a Euclidean space $R^{(n)}$. We solve the problem of seeking the root $\bar{x}$ of the equation $f(x) = 0$ by use of the recurrence procedure

$$x(k+1) = x(k) + a(k)\, y(k+1),\ \ x(0) = x_0,\ \ k = 0,\ldots \tag{1.1}$$

The random n-vector $y(k+1)$ defined at each step of the equality

$$y(k+1) = f(k,\ x(k),\ \eta(k+1)) + \sigma(k,\ x(k))\, \xi(k+1) \tag{1.2}$$

is accessible to measurement. Here $\eta(k)$ is a Markov sequence with an arbitrary bounded set of states $\eta(k) \in Y$. The noise $\xi(k)$, $k = 1, \ldots$ form a sequence of independent r-dimensional vectors, also independent of $\eta(k)$, and

$$M\xi_k = 0,\ M\xi_k\xi_k' = E_r$$

In the notation adopted, M is the symbol for the mean, the prime denotes transposition, and E_r is the unit r-matrix. The matrix $\sigma(k, x)$, in general, is unknown, the dependence of $f(k, x, \eta)$ on η is made concrete later, and $a(k)$ is a nonnegative number sequence. Under these conditions we are required to find constraints on the random process $\eta(k)$ and on the functions $f(x)$, $f(k, x, \eta)$, $\sigma(k, x)$ and $a(k)$, which ensure the convergence $x(k) \to 0$ as $k \to \infty$ with probability 1.

The following model can serve as a natural generalization of procedure (1.1), (1.2) to the case of continuous measurements. Let the n-vector-valued signal $y(t)$, $t \geqslant 0$, defined by

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$$y(t) = f(t, x(t), \eta(t)) + \sigma(t, x(t))\, \xi^{\cdot} \tag{1.3}$$

be accessible to measurement. To determine the root of the equation $f(x) = 0$ we construct the continuous procedure

$$dx = a(t)\,[f(t), x(t), \eta(t))\, dt + \sigma(t, \bar{x}(t))\, d\xi],\ x(0) = x_0,\ \eta(0) = \eta_0 \tag{1.4}$$

As in the discrete case, we are to find concrete constraints on the parameters of system (1.4), under which $x(t) \to \bar{x}$ as $t \to \infty$ with probability 1. We limit the discussion only to the problem's continuous version since the results for the discrete case are formulated analogously. Since the function $f(x)$ has a single zero $\bar{x}$, without loss of generality we can take it that $\bar{x} = 0$.

2. Convergence of the Robbins-Monroe procedure in the presence of purely discontinuous Markov perturbations. We examine the procedure of seeking the root of the equation $f(x) = 0$, setting $f(t, x, \eta) = f(x) + g(t, x, \eta)$ in (1.4), i.e., we consider the system

$$dx = a(t)\,[(f(x) + g(t, x, \eta))\, dt + \sigma(t, x)\, d\xi] \tag{2.1}$$

We assume that the scalar Markov process $\eta(t)$ is purely discontinuous, having a compact set of states $\eta(t) \in Y$ and admitting of the decomposition /11/

$$P\{\eta(\tau) = \alpha,\ t \leqslant \tau \leqslant t + \Delta t \mid \eta(t) = \alpha\} = 1 - q(\alpha)\,\Delta t + o(\Delta t) \tag{2.2}$$

$$P\{\eta(t + \Delta t) \neq \alpha,\ \eta(t + \Delta t) \in G \mid \eta(t) = \alpha\} = q(\alpha, G)\,\Delta t + o(\Delta t),\ \alpha \notin G$$

where $P\{A \mid B\}$ is the conditional probability. Under these conditions the relations (2.1) and (2.2) and the initial conditions

$$x(t_0) = x_0,\quad \eta(t_0) = \eta_0 \tag{2.3}$$

define on set $\{t \geqslant t_0\} \times R^{(n)} \times Y$ a random Markov process $\{x(t), \eta(t)\}$ a separable modification of which has the continuous realizations $x(t, \omega)$ and the right-continuous realizations $\eta(t, \omega)$. Let us state the conditions ensuring the convergence of procedure (2.1), (2.2).

1°. The vector-valued function $f(x)$ is defined on $R^{(n)}$ and its components have bounded first- and second-order partial derivatives

$$\left|\frac{\partial f_i}{\partial x_j}\right| \leqslant L,\quad \left|\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right| \leqslant L,\qquad i, j, k = 1, \ldots, n \tag{2.4}$$

2°. The system $\dot{x} = f(x)$ is exponentially stable in-the-large and, consequently /12/, a positive-definite scalar function $v(x)$ exists admitting of the estimate

$$c_1 \|x\|^2 \leqslant v(x) \leqslant c_2 \|x\|^2,\ (\partial v/\partial x)' f(x) \leqslant -c_3 \|x\|^2 \tag{2.5}$$

$$\|\partial v/\partial x\| \leqslant c_4 \|x\|,\ \|\partial^2 v/\partial x^2\| \leqslant c_5 \tag{2.6}$$

Here $\partial v/\partial x$ is the vector with coordinates $\partial v/\partial x_i$, $\partial^2 v/\partial x^2$ is an $n \times n$-matrix comprised of the second derivatives $\partial^2 v/\partial x_i \partial x_j$, $c_1, \ldots, c_5$ are positive constants.

3°. The estimate $\|g(t, x, \eta)\| \leqslant \varphi(\eta)\|x\|$ is valid, where the function $\varphi(\eta)$ is bounded on Y by a number $M > 0$, and for some $\gamma > 0$

$$S = \{\eta : c_4 \varphi(\eta) \leqslant c_3 - \gamma\} \neq \varnothing \tag{2.7}$$

4°. The intensity of the white noise $\sigma(t, x)$ is bounded in norm

$$\|\sigma(t, x)\,\sigma'(t, x)\| \leqslant K(1 + \|x\|^2),\ K > 0,\ \|A\|^2 = \operatorname{Tr}(AA') \tag{2.8}$$

5°. The differentiable function $a(t), t \geqslant 0$ is monotonic and nonnegative, and

$$\int_0^{\infty} a(t)\, dt = \infty,\qquad \int_0^{\infty} a^2(t)\, dt < \infty \tag{2.9}$$

The following statement is valid:

Theorem 2.1. Let conditions $1^\circ - 5^\circ$ be fulfilled. Then we can find numbers $q_1 > 0$, $q_2 > 0$ such that the inequalities

$$q(\alpha, V) < q_1, \quad \alpha \in S; \quad q(\beta, S) > q_2, \quad \beta \in V = Y \setminus S \tag{2.10}$$

ensure the convergence $x(t) \to 0$ as $t \to \infty$ with probability 1 under any initial conditions (2.3).

Proof. We consider the function /13/

$$V(t, x, \eta) = \begin{cases} v(x), & \eta \in S \\ (1 + \mu a(t)) v(x), & \eta \in V \end{cases} \tag{2.11}$$

where μ is some positive constant which we deal with later on. This function is positive definite, admits of an infinite lower bound and an infinitesimal upper bound in x in the domain $\{t \geqslant 0\} \times R^{(n)} \times Y$, and has a continuous partial derivative in t and bounded first and second derivatives in x. Its averaged derivative /4/ relative to system (2.1) at the point (x, η, t) is computed by the formula

$$\frac{dM[V]}{dt} = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)' (f(x) + g(t, x, \eta)) a(t) + \tag{2.12}$$

$$\frac{a^2(t)}{2} \operatorname{Tr}\left(\frac{\partial^2 V}{\partial x^2} \sigma\sigma'\right) + \int [V(t, x, \vartheta) - V(t, x, \eta)] q(\eta, d\vartheta)$$

Let us estimate $dM[V]/dt$. On the strength of (2.5)—(2.8) we obtain:

at point $t \geqslant t_0$, $x \in R^{(n)}$, $\eta \in S$

$$\frac{dM[V]}{dt} \leqslant -a(t)[\gamma(t) - \mu c_2 q(\eta, V)] \|x\|^2 + \frac{1}{2} K c_5 a^2(t)(1 + \|x\|^2)$$

at point $t \geqslant t_0$, $x \in R^{(n)}$, $\eta \in V$

$$\frac{dM[V]}{dt} \leqslant a-(t)[\mu c_1 q(\eta, S) - (1 + \mu a(t))(c_4 M - c_3)] \|x\|^2 + \frac{1}{2} K c_5 (1 + \mu a(t)) a^2(t)(1 + \|x\|^2)$$

Now if the inequalities

$$q(\eta, V) < \gamma \mu c_2^{-1}, \quad q(\eta, S) > (1 + \mu a(0))(c_4 M - c_3)(\mu c_1)^{-1}$$

are fulfilled for some $\mu > 0$, then the condition

$$\frac{dM[V]}{dt} \leqslant -a(t)\delta\|x\|^2 + h(t)(1 + V(t, x, \eta)) \tag{2.13}$$

is fulfilled for any $x \in R^{(n)}, \eta \in Y, t \geqslant t_0$, where $\delta > 0$ is some constant, $h(t)$ is a function integrable on $[0, \infty)$. The rest of the proof is along a known plan (/2/, p.100) if as the set B figuring in /2/ we take $B = \{(x, \eta) : x = 0, \eta \in Y\}$ which in the case at hand is invariant /5/ for the process $\{x(t), \eta(t)\}$.

Note 2.1. The probabilistic sense of the theorem just proved is that under the conditions stated the stochastic approximation procedure converges almost surely to the zero of function $f(x)$ if the probabilities of the transitions in time Δt from small values of perturbations $g(t, x, \eta)$ to large ones are sufficiently small, while the inverse transition probabilities are sufficiently large. In addition, it should be stressed that under the conditions the variations of process $\eta(t)$, taking place within sets S and V, do not affect the convergence of procedure (2.1).

Note 2.2. Condition 2° of Theorem 2.1 can be relaxed, requiring only the asymptotic stability in-the-large of system $x^\cdot = f(x)$, but then the constraint (2.8) on the white noise's intensity must be expressed in terms of the parameters of the Liapunov function.

We now consider a one-dimensional system

$$dx = a(t)((f(x) + g(t, x, \eta)) dt + \sigma(t, x) d\xi) \tag{2.14}$$

We assume that the unknown functions $f(x)$, $g(t, x, \eta)$, $\sigma(t, x)$ satisfy the conditions

$$xf(x) \leqslant -c_3x^2, \quad |g(t, x, \eta)| \leqslant \varphi(\eta)\,|x|, \qquad \sigma^2(t, x) \leqslant K(1 + x^2) \tag{2.15}$$

and that the random Markov process $\eta(t)$ can be in only the two states η_1 and η_2. We accept that $\varphi(\eta_1) - c_3 < 0$ and $\varphi(\eta_2) - c_3 > 0$, otherwise the problem becomes trivial. We denote the expansion coefficients in (2.2) by q_{12} and q_{21}, respectively. Choosing $v(x) = {}^1/_2 x^2$, we construct the function $V(t, x, \eta)$ in form (2.11). For $dM[V]/dt$ to satisfy bound (2.13) it is sufficient to require the fulfillment of the conditions

$$\varphi(\eta_1) - c_3 + \mu q_{12} \leqslant -\varepsilon, \quad (1 + \mu a(t))(\varphi(\eta_2) - c_3 - \mu q_{21}) \leqslant -\varepsilon \tag{2.16}$$

for some positive values of μ and ε. Let the inequality

$$\beta = (\varphi(\eta_1) - c_3)\, q_{21} + (\varphi(\eta_2) - c_3)\, q_{12} < 0 \tag{2.17}$$

be fulfilled; then we can find an instant $T \geqslant t_0$ and values $\mu > 0$, $\varepsilon > 0$ so small that when $t > T$ conditions (2.16) are valid and, consequently, $x(t) \to 0$ as $t \to \infty$ with probability 1. Thus, the fulfillment of (2.17) is sufficient for the convergence of procedure (2.14). We should emphasize that this condition cannot be weakened since for the linear equation $x^{\cdot} = -c_3x + \varphi(\eta)\,x$ it is a necessary and sufficient condition for the convergence of $x(t)$ to zero /14/ with probability 1.

3. Convergence of the stochastic approximation procedure in the presence of continuous Markov noise. Let us examine the procedure of seeking the zero of an unknown function $f(x)$ under the condition that its measurements contain, together with Gaussian white noise, continuous Markiov perturbations as well, which are modelled as the output signals of an asymptotically stable system. In other words, let the zero seeking procedure for function $f(x)$ be described by a system of Itô stochastic differential equations

$$\begin{aligned} dx &= a(t)\,[(f(x) + A\eta)\,dt + \sigma_1(t, x)\,d\xi_1] \\ d\eta &= B\eta dt + \sigma_2(t, x)\,d\xi_2, \quad x(t_0) = x_0, \quad \eta(t_0) = \eta_0 \end{aligned} \tag{3.1}$$

Here vectors x and η are of dimensions n and m, respectively, $f(x)$, A, B, $\sigma_1(t, x)$, $\sigma_2(t, x)$ are matrices unknown in general, of appropriate dimensions, $\xi_1(t)$, $\xi_2(t)$ are independent standard Wiener processes of dimensions r and s. We formulate below the conditions under which procedure (3.1) ensures the convergence $x(t) \to 0$ as $t \to \infty$ with probability 1.

Theorem 3.1. Let the functions $f(x)$, $\sigma_i(t, x)$ $(i = 1, 2)$, $a(t)$ satisfy conditions (2.4)—(2.6), (2.8), (2.9) and, in addition, let the eigenvalues of matrix B have negative real parts, while the function $a(t)$ satisfies, for some value of constant $\delta > 0$, the inequality $|a^{\cdot}(t)\, a^{-2}(t)| \leqslant \delta$. Then under any initial conditions $x(t_0) = x_0$, $\eta(t_0) = \eta_0$ the equality

$$P\{\lim_{t\to\infty} x(t) = 0 \mid x(t_0) = x_0, \quad \eta(t_0) = \eta_0\} = 1$$

is valid for the solution $x(t)$.

Proof. We take a function $v(x)$ satisfying conditions (2.5) and (2.6) and we construct a quadratic form $w(\eta)$ for which the estimate

$$\begin{gathered} e_1\|\eta\|^2 \leqslant w(\eta) \leqslant e_2\|\eta\|^2, \ (\partial w/\partial\eta)' B\eta \leqslant -e_3\|\eta\|^2 \\ \|\partial w/\partial\eta\| \leqslant e_4\|\eta\|, \ \|\partial^2 w/\partial\eta^2\| \leqslant e_5 \end{gathered}$$

are valid, where $e_1, \ldots, e_5$ are positive constants. (The latter is possible by virtue of the assumptions on the properties of matrix B). Passing in system (3.1) to the new variables /8/

$$z = a(t)\eta, \quad y = x - AB^{-1}z$$

we obtain the system

$$\begin{aligned} dy &= a(t)[(f(y + AB^{-1}z) + AB^{-1}a^{\cdot}(t)a^{-1}(t)z)dt + \sigma_1 d\xi_1 + AB^{-1}\sigma_2 d\xi_2] \\ dz &= (Bz - a^{\cdot}(t)z)dt + a(t)\sigma_2(t, x)d\xi_2 \end{aligned} \tag{3.2}$$

We take the function $V(y, z) = v(y) + \mu w(z)$, where $\mu > 0$ is some constant. Computing $dM[V]/dt$ relative to system (3.2) with due regard to the theorem's conditions, we obtain

$$dM\,[V]/dt \leqslant -a\,(t)c_3\,\|\,y\,\|^2 + a\,(t)c_4\,(L\ + \delta)\,\|\,AB^{-1}\,\|\,\|\,y\,\|\,\|\,z\,\| - \mu\,(e_3 - e_4 a\,(t)\delta)\,\|\,z\,\|^2 +$$
$$a^2\,(t)(Kc_5 + {}^1/{}_2\mu e_5)(1 + \|\,y + AB^{-1}z\,\|^2)$$

Hence by simple but cumbersome manipulations we get that starting from some instant $T \geqslant t_0$ the estimate

$$dM\,[V]/dt \leqslant -\alpha\,(t)\varphi\,(y,\,z) + h\,(t)(1 + V\,(y,\,z)) \tag{3.3}$$

is valid under an appropriate choice of number μ, where the functions $\alpha\,(t) > 0,\ h\,(t) > 0$ satisfy the conditions

$$\int_T^\infty \alpha\,(t)\,dt = \infty, \qquad \int_T^\infty h\,(t)\,dt < \infty$$

and $\varphi\,(y,\,z)$ is a positive-definite form in $R^{(n)} \times R^{(m)}$. From here on we make use of Theorem 8.1 from /2/. Thus, $y\,(t) \to 0,\ z\,(t) \to 0$ as $t \to \infty$ with probability 1, which completes the theorem's proof.

Note 3.1. The procedure described can be used for seeking the minimum of a positive-definite scalar function $F\,(x),\ x \in R^{(n)}$, it if is possible to measure $\partial F/\partial x$ with noise represented both as a Markov component as well as a white noise independent of it. In this case we should set $f\,(x) = -\partial F/\partial x,\ v\,(x) = F\,(x)$ in the preceding considerations. As a result we obtain convergence conditions for the gradient procedure, close to those examined in /8/ for the discrete case.

In conclusion we discuss very briefly the case of measurements continuous in time in the presence of random jumps. Let the zero seeking procedure for function $f\,(x)$ be described by stochastic differential equations with jumps /4,6/

$$dx = a\,(t)[(f\,(x) + A\eta)dt + \sigma_1\,(t,\,x)d\xi_1 + g_1\,(t,\,x)d\zeta_1]$$
$$d\eta = B\eta dt + \sigma_2\,(t,\,x)d\xi_2 + g_2\,(t,\,x)d\zeta_2$$

They differ from Eqs.(3.1) only in that, together with the independent standard Wiener processes $\xi_1\,(t),\ \xi_2\,(t)$, in them occur $\zeta_1\,(t),\ \zeta_2\,(t)$, namely, Poisson processes independent of each other and of $\xi_1\,(t),\ \xi_2\,(t)$, with the probability $\lambda_i\Delta t + o\,(\Delta t)$ that a jump takes place in process $\zeta_i\,(t)$ on the interval $[t,\,t + \Delta t]$. Under the condition that the jump took place, we denote by $P_i\,(du)$ the corresponding probability measure of the jump's amplitude. We take it that $P_i\,(du)$ has a compact support U and that

$$\int_U uP_i\,(du) = 0, \quad \int_U u^2P_i\,(du) = \nu_i^2 < \infty$$

These equations and the initial conditions $x\,(t_0) = x_0,\ \eta\,(t_0) = \eta_0$ define a process $\{x\,(t),\ \eta\,(t)\}$ whose realizations are right-continuous with probability 1 (the existence and uniqueness conditions for the solution of such equations are given in /6/).

If we retain all the assumptions in Theorem 2.1 and require that the condition

$$\|\,g_i\,(t,\,x)g_i'\,(t,\,x)\,\| \leqslant K\,(1 + \|\,x\,\|^2), \quad i = 1,\,2 \tag{3.4}$$

be fulfilled, then we can repeat all the arguments of this theorem's proof. As a matter of fact, the only difference will be the appearance of additional summands of the form

$$\int_U [v\,(x + a\,(t)\,g_i\,(t,\,x)\,u) - v\,(x)]\,\lambda_i P_i\,(du)$$

when computing $dM[V]/dt$. However, in the presence of a bounded second derivative $\partial^2 v/\partial x^2$ these summands are majorized by the function $c_6\nu_i^2\lambda_i a^2\,(t)\,(1 + v\,(x))$ and the final estimate of $dM[V]/dt$ will have the form (3.3) as before. We can convince ourselves that $\{x\,(t),\,\eta\,(t)\}$ is a Feller process and, hence, since its trajectories are right-continuous, it has the strict Markov property. This enables us to make deductions on its regularity and reflexivity with respect to the domain $\|x\| < \varepsilon$ for any $\varepsilon > 0$. The subsequent arguments are along the lines of the proof of the corresponding theorem for the continuous case. We note that an analogous jump-like component can be added on in Eq.(2.1). If in that case a condition of type (3.4) is fulfilled, then the conclusion of Theorem 2.1 remains valid.

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